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New solutions of the Dirac equation*

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ABSTRACT. An effective approach for solving the three-dimensional Dirac equation for spherically symmetric local interactions, which we have introduced recently, is reviewed. The merit of the approach is in producing Schrödinger-like equation for the spinor components that could simply be solved by correspondence with well-known exactly solvable non-relativistic problems. Taking the nonrelativistic limit reproduces the nonrelativistic problem. The approach has been used successfully in establishing the relativistic extension of all classes of shape invariant potentials as well as other exactly solvable nonrelativistic problems. These include the Coulomb, Oscillator, Scarf, Pöschl-Teller, Woods-Saxon, etc.

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I. Introduction

In recent decades the scientific community made remarkable progress in the advancement of human knowledge and in the technological achievements by working relentlessly in large collaborations on the solutions of complicated realistic problems. Examples of such accomplishments in physics are evident in the unprecedented growth of findings and discoveries in a wide range of fields including material science, elementary particles, and information theory. Consequently, one may choose not to disagree with the view that working, as individuals or in very small groups, on much simpler (although, fundamental) problems like exact solutions of the wave equation might not be rewarding. Moreover, it is sometimes argued that such exact solutions are by some (debatable) definitions "trivial". Nevertheless, it is a fact that exact solutions are important because of the conceptual understanding of physics that can only be brought about by the analysis of such solutions. Many of us, including some of our undergraduate students, appreciate the lessons we all learned from exact solutions of simple problems like the harmonic oscillator and the Coulomb problem. Such lessons include, but not limited to: the vivid understanding of the uncertainty principle, creation and annihilation of states, correspondence

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principle, wave-particle duality, energy of the vacuum, spectrum degeneracy, chemical bonds, etc. In fact, exact solutions are valuable means for checking and improving models and numerical methods being introduced for solving complicated physical problems. Furthermore, in some limiting cases or for some special circumstances they may constitute analytic solutions of realistic problems or approximations thereof. In nonrelativistic quantum mechanics, the search for exact solutions of the wave equation was carried out over the years by many authors where several classes of these solvable potentials are accounted for and tabulated (see, for example, the references cited in [1]). Most of the known exactly solvable problems fall within distinct classes of what is referred to as "shape invariant potentials". Supersymmetric quantum mechanics [2], potential algebras [3], and "point canonical transformations" [4] are three methods among many which are used in the search for exact solutions of the wave equation. These developments were extended to other classes of conditionally exactly [5] and quasi exactly [6] solvable problems where all or, respectively, part of the energy spectrum is known.

The relativistic extension of these formulations, on the other hand, remained for a long time only partially developed. Despite all the work that has been done over the years on the Dirac equation, its exact solution for local interaction has been limited to a very small set of potentials. Since the original work of Dirac, in the early part of last century, up until 1989 only the relativistic Coulomb problem was solved exactly. In 1989, the relativistic extension of the oscillator problem (Dirac-Oscillator) was finally formulated and solved by Moshinsky and Szczepaniak [7]. Recently, and in a series of articles [8-11], we presented an effective approach for solving the three dimensional Dirac equation for spherically symmetric potential interaction. The first step in the program started with the realization that the nonrelativistic Coulomb, Oscillator, and S-wave Morse problems belong to the same class of shape invariant potentials which carries a representation of SO(2,1) Lie algebra. The solutions of these problems could be mapped into one another by point canonical transformations. Therefore, the fact that the relativistic version of the first two problems (Dirac-Coulomb and Dirac-Oscillator) were solved exactly makes the solution of the third, in principle, feasible. Indeed, the relativistic s-wave Dirac-Morse problem was formulated and solved in Ref. 8. The bound state energy spectrum and spinor wavefunctions were obtained. Taking the nonrelativistic limit reproduces the familiar s-wave Schrödinger-Morse problem. Motivated by these findings, the same approach was applied successfully in obtaining solutions for the relativistic extension of yet another class of shape invariant potentials [9]. These included the Dirac-Scarf, Dirac-Rosen-Morse I & II, Dirac-Pöschl-Teller, and Dirac-Eckart potentials. Furthermore, using the same formalism quasi exactly solvable systems at rest mass energies were obtained for a large class of power-law relativistic potentials [10]. Quite recently, Guo Jian-You *et al* succeeded in constructing solutions for the relativistic Dirac-Woods-Saxon and Dirac-Hulthén problems using the same approach [12]. In the fourth and last article of the series in our program of searching for exact solutions to the Dirac equation [11], we found a special graded extension of SO(2,1) Lie algebra. Realization of this superalgebra by 2×2 matrices of differential operators acting in the two component spinor space was constructed. The linear span of this graded algebra gives the canonical form of the radial Dirac Hamiltonian. It turned out that the Dirac-Oscillator class, which also includes the Dirac-Coulomb and Dirac-Morse, carries a representation of this supersymmetry.

The central idea in the approach is to separate the variables such that the two coupled first order differential equations resulting from the radial Dirac equation generate

Schrödinger-like equation. This makes the solution of the relativistic problem easily attainable by simple and direct correspondence with well-known exactly solvable nonrelativistic problems. The correspondence results in a map among the relativistic and nonrelativistic parameters. Using this map and the known nonrelativistic energy spectrum one can easily and directly obtain the relativistic spectrum. Moreover, the two components of the spinor wavefunction are obtained from the nonrelativistic wavefunction using the same parameter map. There are two main ingredients in the formulation of the approach that makes it work. The first is a unitary transformation of the Dirac equation which, of course, reduces to the identity in the nonrelativistic limit. The second is the introduction, in a natural way, of an auxiliary potential component which is constrained to depend, in a particular way, on the independent potential function of the problem.

The approach is initiated by writing the relativistic Hamiltonian for a Dirac spinor coupled "non-minimally" to a four-component potential $(A_0, \vec{A})$. Spherical symmetry is imposed on the interaction by restricting the potential to the form $(A_0, \vec{A}) = [V(r), \hat{r}cW(r)]$, where c is the speed of light and $\hat{r}$ is the radial unit vector. $V(r)$ and $W(r)$ are real radial functions referred to as the even and odd components of the relativistic potential, respectively. The resulting Dirac equation gives two coupled first order differential equations for the two radial spinor components. By eliminating one component we obtain a second order differential equation for the other. The resulting equation may turn out not to be Schrödinger-like, i.e. it may contain first order derivatives. Obtaining a Schrödinger-like wave equation is desirable because it results in a substantial reduction of the efforts needed for getting the solution. It puts at our disposal a variety of well established techniques to be employed in the analysis and solution of the problem. These techniques have been well developed over the years by many researchers in dealing with the Sturm-Liouville problem and the Schrödinger equation. One such advantage, which will become clear shortly, is the resulting map between the parameters of the relativistic and nonrelativistic problems. This parameter map could be used in obtaining, for example, the relativistic energy spectrum in a simple and straight-forward manner from the known nonrelativistic spectrum. A global unitary transformation is applied to the Dirac equation to eliminate the first order derivative. The Schrödinger-like requirement produces a constraint in the form of a linear relation between the two potential components as $V \sim W + \kappa/r$, where κ is the spin-orbit quantum number defined as $\kappa = \pm(j+½) = \pm1, \pm2, \ldots$ for $\ell = j \pm ½$. This will result in a Hamiltonian that will be written in terms of only one arbitrary potential function: either $V(r)$ or $W(r)$. Meeting the Schrödinger-like requirement is generally possible only because of the degree of flexibility brought about by the presence of an auxiliary potential component. However, the unitary transformation is not necessary when $V = 0$. This corresponds to the case of the superpotentials $U^2 \pm U'$. The Dirac-Oscillator, where $W \sim r$, is an example of such a case. The Dirac-Scarf and Dirac-Pöschl-Teller potentials are also two among other such examples.

The paper is organized as follows. In Sec. II, we set up the three dimensional Dirac equation for a spinor coupled in a non-minimal way to the four-potential $(A_0, \vec{A})$. Spherical symmetry is imposed reducing the problem to a solution of the 2×2 radial component of the Dirac equation. An overview of the technical details of the formalism will be presented in Sec. III. Implementation of the approach on selected potential examples will be given in the same section, while a comprehensive list will be displayed in a tabular form.

II. The three-dimensional dirac equation

Dirac equation is a relativistically covariant first order differential equation in four dimensional space-time for a spinor wavefunction ψ. For a free structureless particle it reads $\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi = 0$, where m is the rest mass of the particle and c is the speed of light. The summation convention over repeated indices is used. That is, $\gamma^\mu\partial_\mu \equiv \sum_{\mu=0}^{3}\gamma^\mu\partial_\mu = \gamma^0\partial_0 + \vec{\gamma}.\vec{\partial} = \gamma^0\frac{\partial}{c\partial t} + \vec{\gamma}.\vec{\nabla}$. $\left\{\gamma^\mu\right\}_{\mu=0}^{3}$ are four constant square matrices satisfying the anticommutation relation $\left\{\gamma^\mu, \gamma^\nu\right\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\mathcal{G}^{\mu\nu}$, where $\mathcal{G}$ is the metric of Minkowski space-time which is equal to $\text{diag}(+,-,-,-)$. These are unimodular even dimensional matrices with a minimum dimension of four corresponding to spin ½ representation of the Lorentz space-time symmetry group. A four-dimensional matrix representation that satisfies the anticommutation relation is chosen as follows:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{2.1}$$

where I is the 2×2 unit matrix and $\vec{\sigma}$ are the three 2×2 hermitian Pauli matrices. Now, if the Dirac particle carries an electric charge e, then it will couple to the four-component electromagnetic potential $\mathcal{A}_\mu = (\mathcal{A}_0, \vec{\mathcal{A}})$. Gauge invariant coupling is accomplished by the "minimal" substitution $\partial_\mu \to \partial_\mu + i\frac{e}{\hbar c}\mathcal{A}_\mu$. It transforms the free Dirac equation to

$$\left[i\hbar\gamma^\mu(\partial_\mu + i\tfrac{e}{\hbar c}\mathcal{A}_\mu) - mc\right]\psi = 0 \tag{2.2}$$

where ψ is a four-component spinor. When written in details, this equation reads

$$i\hbar\gamma^0\partial_0\psi = \left(-i\hbar\vec{\gamma}\cdot\vec{\nabla} + \tfrac{e}{c}\vec{\gamma}\cdot\vec{\mathcal{A}} + \tfrac{e}{c}\gamma^0\mathcal{A}_0 + mc\right)\psi \tag{2.3}$$

Multiplying both sides by $c\gamma^0$ gives

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-i\hbar c\vec{\alpha}\cdot\vec{\nabla} + e\vec{\alpha}\cdot\vec{\mathcal{A}} + e\mathcal{A}_0 + mc^2\beta\right)\psi \tag{2.4}$$

where $\vec{\alpha}$ and β are the hermitian matrices

$$\vec{\alpha} = \gamma^0\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \text{ and } \beta = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{2.5}$$

For time independent potentials, equation (2.4) gives the following matrix representation of the Dirac Hamiltonian (in units of mc^2)

$$H = \begin{pmatrix} \frac{e}{mc^2}\mathcal{A}_0 + 1 & -i\frac{\hbar}{mc}\vec{\sigma}\cdot\vec{\nabla} + \frac{e}{mc^2}\vec{\sigma}\cdot\vec{\mathcal{A}} \\ -i\frac{\hbar}{mc}\vec{\sigma}\cdot\vec{\nabla} + \frac{e}{mc^2}\vec{\sigma}\cdot\vec{\mathcal{A}} & \frac{e}{mc^2}\mathcal{A}_0 - 1 \end{pmatrix} \tag{2.6}$$

$$\begin{pmatrix} +1+\lambda^2 V(r)-\varepsilon & \lambda\left[\frac{\kappa}{r}+W(r)-\frac{d}{dr}\right] \\ \lambda\left[\frac{\kappa}{r}+W(r)+\frac{d}{dr}\right] & -1+\lambda^2 V(r)-\varepsilon \end{pmatrix}\begin{pmatrix} g(r) \\ f(r) \end{pmatrix}=0 \tag{2.11}$$

This results in two coupled first order differential equations for the two radial spinor components. In the following section we show how to separate the variables such that the resulting second order differential equation is easily solvable.

III. Transformation of the dirac equation and its solution

In Eq. (2.11), eliminating one spinor component in favor of the other gives a second order differential equation. This will not be Schrödinger-like (i.e., it contains first order derivatives) unless V = 0. To obtain a Schrödinger-like equation in the general case we proceed as follows. A global unitary transformation $\mathcal{U}(\eta)=\exp(\frac{i}{2}\lambda\eta\sigma_2)$ is applied to the Dirac equation (2.1), where η is a real constant parameter and σ_2 is the 2×2 Pauli matrix $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. The Schrödinger-like requirement relates the two potential components by the linear constraint $V(r)=\xi\left[W(r)+\kappa/r\right]$, where ξ is a real parameter. It also requires that $\sin(\lambda\eta)=\pm\lambda\xi$, where $-\frac{\pi}{2}<\lambda\eta<+\frac{\pi}{2}$. This results in a Hamiltonian that will be written in terms of only one arbitrary potential function; either the even potential component $V(r)$ or the odd one $W(r)$. Moreover, the solution of the problem is obtained for a given value of κ. It is to be noted that the angular parameter of the unitary transformation $\mathcal{U}(\eta)$ was intentionally split as $\lambda\eta$ and not collected into a single angle, say φ. This is suggested by investigating the constraint $\sin(\varphi)=\pm\lambda\xi$ in the nonrelativistic limit ($\lambda\rightarrow 0$) where we should have $\sin(\varphi)\approx\varphi=\pm\lambda\xi$. It also makes it obvious that in the nonrelativistic limit the transformation becomes the identity (i.e., not needed).

The unitary transformation together with the potential constraint map Eq. (2.11) into the following one, which we choose to write in terms of the odd potential component

$$\begin{pmatrix} C-\varepsilon+(1\pm 1)\lambda^2\xi\left(W+\frac{\kappa}{r}\right) & \lambda\left[\mp\xi+C\left(W+\frac{\kappa}{r}\right)-\frac{d}{dr}\right] \\ \lambda\left[\mp\xi+C\left(W+\frac{\kappa}{r}\right)+\frac{d}{dr}\right] & -C-\varepsilon+(1\mp 1)\lambda^2\xi\left(W+\frac{\kappa}{r}\right) \end{pmatrix}\begin{pmatrix} \phi^{+}(r) \\ \phi^{-}(r) \end{pmatrix}=0 \tag{3.1}$$

where $C=\cos(\lambda\eta)=\sqrt{1-(\lambda\xi)^2}>0$ and

$$\begin{pmatrix} \phi^{+} \\ \phi^{-} \end{pmatrix}=\mathcal{U}\psi=\begin{pmatrix} \cos\frac{\lambda\eta}{2} & \sin\frac{\lambda\eta}{2} \\ -\sin\frac{\lambda\eta}{2} & \cos\frac{\lambda\eta}{2} \end{pmatrix}\begin{pmatrix} g \\ f \end{pmatrix} \tag{3.2}$$

Equation (3.1) gives the following equation for one spinor component in terms of the other

$$\phi^{\mp}(r)=\frac{\lambda}{C\pm\varepsilon}\left[-\xi\pm C\left(W+\frac{\kappa}{r}\right)+\frac{d}{dr}\right]\phi^{\pm}(r) \tag{3.3}$$

While, the resulting Schrödinger-like wave equation becomes

The wave equation reads $(H-\varepsilon)\psi=0$, where ε is the relativistic energy which is real and measured in units of mc^2. We choose units where $\hbar = m = e = 1$. In these units the relativistic parameter is the Compton wavelength $\lambda=\hbar/mc=c^{-1}$. It should be noted that in these units the role of the fine structure constant is played by the Compton wavelength λ. The units ($\hbar = c = 1$) where the fine structure constant is used as the relativistic parameter are suitable for the electromagnetic interaction. The units that we are adopting here, where the relativistic parameter is λ, are suitable for dealing with a larger class of problems.

Now, the space component $\vec{A}$ of the electromagnetic potential could be eliminated by the usual electromagnetic gauge transformation. Consequently, the contribution of the off-diagonal term $\vec{\sigma}\cdot\vec{A}$ could be gauged away when the Dirac particle is minimally coupled to the electromagnetic potential $(A_0,\vec{A})$. However, in our approach the spinor is coupled in a non-minimal way to a four-vector potential $(A_0,\vec{A})$. This is accomplished by replacing the two off-diagonal terms $\vec{\sigma}\cdot\vec{A}$ in the above Hamiltonian (2.6) by $\pm i\vec{\sigma}\cdot\vec{A}$, respectively. That is the Hamiltonian, in our chosen units, is replaced by the following

$$H=\begin{pmatrix} \lambda^2 A_0+1 & -i\lambda\vec{\sigma}\cdot\vec{\nabla}+i\lambda^2\vec{\sigma}\cdot\vec{A} \\ -i\lambda\vec{\sigma}\cdot\vec{\nabla}-i\lambda^2\vec{\sigma}\cdot\vec{A} & \lambda^2 A_0-1 \end{pmatrix} \qquad (2.7)$$

It should be noted that this type of coupling does not support an interpretation of $(A_0,\vec{A})$ as the electromagnetic potential unless, of course, $\vec{A}=0$ (e.g., the Coulomb potential). Likewise, H does not have local gauge symmetry. That is, the associated wave equation is not invariant under the usual electromagnetic gauge transformation. Now, we impose spherical symmetry and write $(A_0,\vec{A})$ as $[V(r),\frac{1}{\lambda}\hat{r}W(r)]$, where $\hat{r}$ is the radial unit vector. In this case, the angular variables could be separated and we can write the spinor wavefunction as [13]

$$\psi=\begin{pmatrix} i[g(r)/r]\chi_{\ell m}^{j} \\ [f(r)/r]\vec{\sigma}\cdot\hat{r}\chi_{\ell m}^{j} \end{pmatrix} \qquad (2.8)$$

where f and g are real radial square-integrable functions. The angular wavefunction for the two-component spinor is written as

$$\chi_{\ell m}^{j}(\hat{r})=\frac{1}{\sqrt{2\ell+1}}\begin{pmatrix} \sqrt{\ell\pm m+1/2}\; Y_{\ell}^{m-1/2} \\ \pm\sqrt{\ell\mp m+1/2}\; Y_{\ell}^{m+1/2} \end{pmatrix}, \qquad \text{for } j=\ell\pm\tfrac{1}{2} \qquad (2.9)$$

$Y_{\ell}^{m\pm 1/2}(\hat{r})$ is the spherical harmonic function and m stands for the half-integers in the range $-j,-j+1,\ldots,j$ and should not be confused with the mass. Spherical symmetry gives $i\vec{\sigma}\cdot(\vec{r}\times\vec{\nabla})\psi(r,\hat{r})=-(1+\kappa)\psi(r,\hat{r})$, where κ is the spin-orbit quantum number defined as $\kappa=\pm(j+\tfrac{1}{2})$ for $\ell=j\pm\tfrac{1}{2}$. Using this we obtain the following useful relations

$$\begin{aligned} (\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\hat{r})F(r)\chi_{\ell m}^{j}(\hat{r}) &= \left(\frac{dF}{dr}+\frac{1-\kappa}{r}F\right)\chi_{\ell m}^{j} \\ (\vec{\sigma}\cdot\vec{\nabla})F(r)\chi_{\ell m}^{j}(\hat{r}) &= \left(\frac{dF}{dr}+\frac{1+\kappa}{r}F\right)(\vec{\sigma}\cdot\hat{r})\chi_{\ell m}^{j} \end{aligned} \qquad (2.10)$$

Employing these in the wave equation $(H-\varepsilon)\psi=0$ results in the following 2×2 matrix equation for the two radial spinor components

$$\left[-\frac{d^2}{dr^2}+\frac{\gamma(\gamma\pm 1)}{r^2}+2V_\kappa^\pm(r,\varepsilon)-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi^\pm(r)=0 \tag{3.4}$$

where

$$V_\kappa^\pm=\frac{1}{2}C^2W^2\mp\frac{C}{2}\frac{dW}{dr}+C^2\kappa\frac{W}{r}+\xi\varepsilon W+\xi\varepsilon\frac{\kappa}{r} \tag{3.5}$$

and $\gamma=C\kappa=\kappa\sqrt{1-(\lambda\xi)^2}$ plays the role of angular momentum in the relativistic theory. In the nonrelativistic limit ($\lambda\to 0$), $\varepsilon\approx 1+\lambda^2E$ and $C\approx 1-\frac{1}{2}\lambda^2\xi^2$. Therefore, Eq. (3.3) shows that ϕ^+ is the larger of the two relativistic spinor components (i.e., ϕ^+ is the component that survives the nonrelativistic limit, whereas $\phi^-\sim\lambda\phi^+\to 0$). Consequently, if we favor the upper spinor component then our choice of sign in the transformation parameter constraint is the top + sign. That is, we choose $\sin(\lambda\eta)=+\lambda\xi$.

In all relativistic problems that have been successfully tackled so far, Eq. (3.4) is solved by correspondence with well-known exactly solvable nonrelativistic problems. This correspondence results in a parameter map that relates the two problems. Now, if the nonrelativistic problem is exactly solvable, then using this parameter map and the known nonrelativistic energy spectrum one can easily obtain the relativistic spectrum. In fact, the relativistic extension of any known dynamical relationship in the nonrelativistic theory could easily be obtained by this correspondence map. The Green's function, which has a prime significance in the calculation of relativistic processes, is such an example [14]. Moreover, the spinor component wavefunction is also obtained from the nonrelativistic wavefunction using the same parameter map.

An alternative, but equivalent, approach to the one given above is to postulate the one-parameter two-component equation (3.1) as the relativistic wave equation and show that in the nonrelativistic limit ($\lambda\to 0$) the nonrelativistic problem is recovered. However, in this case, one cannot claim that the relativistic problem is a unique extension of the nonrelativistic one.

To illustrate the implementation steps of the approach we will now apply it on some selected examples.

A. The Dirac-Coulomb problem

For this problem $W(r)=0$. Therefore, the Schrödinger-like requirement gives the potential constraint that yields $V(r)=\xi\kappa/r\equiv Z/r$, where Z is the spinor charge. This gives $\xi=Z/\kappa$ relating the transformation parameter to the physical constants of the problem as $\sin(\lambda\eta)=\lambda Z/\kappa=\alpha\mathcal{Z}/\kappa$, where α is the fine structure constant and $\mathcal{Z}$ is the dimensionless spinor charge in units of e. The wave equation (3.4), for the upper spinor component reads

$$\left[-\frac{d^2}{dr^2}+\frac{\gamma(\gamma+1)}{r^2}+2\frac{Z\varepsilon}{r}-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi^+(r)=0 \tag{3.6}$$

where $\gamma = \kappa\sqrt{1-(\lambda Z/\kappa)^2} = \kappa\sqrt{1-(\alpha Z/\kappa)^2}$ is the relativistic angular momentum. Comparing this equation with the nonrelativistic Schrödinger-Coulomb wave equation

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\frac{Z}{r}-2E\right]\Phi(r)=0 \tag{3.7}$$

gives, by correspondence, the following map between the parameters of the two problems:

$$Z \rightarrow Z\varepsilon,\, E \rightarrow \tfrac{\varepsilon^2-1}{2\lambda^2},\, \ell \rightarrow \begin{cases} \gamma \\ -\gamma-1 \end{cases} \tag{3.8}$$

The top (bottom) choice of the ℓ map corresponds to positive (negative) values of κ, respectively. It should be noted that the map produced by the comparison of Eq. (3.6) to Eq. (3.7) is a "correspondence" map between the parameters of the two problems and not an equality of the parameters. That is we obtain, for example, the correspondence map $\ell \rightarrow \gamma$ but not the equality $\ell = \gamma$. In fact, γ is not an integer while, of course, ℓ is. Using the parameter map in (3.8) and the well-known nonrelativistic energy spectrum, E_n = $-Z^2/2(\ell+n+1)^2$, we obtain the following relativistic spectrum for $\kappa > 0$

$$\varepsilon = \varepsilon_n^\gamma = \pm\left[1+\left(\tfrac{\lambda Z}{n+\gamma+1}\right)^2\right]^{-1/2}; \qquad n = 0,1,2,\ldots \tag{3.9}$$

For $\kappa < 0$, the energy spectrum is obtained with $\ell \rightarrow -\gamma-1$. That is, $\varepsilon = \varepsilon_n^{-\gamma-1}$. It is worthwhile noting that the lowest positive energy eigenvalue is $\varepsilon_+ = \left|\varepsilon_0^{-\gamma-1}\right| = \gamma/\kappa$ = $\sqrt{1-(\lambda Z/\kappa)^2}$ for $\kappa = -1,-2,\ldots$ The highest negative energy, on the other hand, is ε_- = $-\left|\varepsilon_0^{-\gamma-1}\right| = -\gamma/\kappa$ for $\kappa = -1,-2,\ldots$ These two correspond to non-degenerate states, while all others do not. That is because $\varepsilon_n^\gamma\Big|_{\kappa>0} = \varepsilon_{n+1}^{-\gamma-1}\Big|_{\kappa<0}$ for $n = 0,1,2,\ldots$ and for all κ: The upper radial component of the spinor wavefunction is obtained using the same parameter map (3.8) in the nonrelativistic wavefunction $\Phi_n(r) \sim (\lambda_n r)^{\ell+1} e^{-\lambda_n r/2} L_n^{2\ell+1}(\lambda_n r)$, where λ_n = $2|Z|/(n+\ell+1)$ and $L_n^\nu(x)$ is the generalized Laguerre polynomial [15]:

$$\phi_n^+(r) = \begin{cases} A_n^\gamma(\omega_n^\gamma r)^{\gamma+1} e^{-\omega_n^\gamma r/2} L_n^{2\gamma+1}(\omega_n^\gamma r) & , \quad \kappa > 0 \\ A_n^{-\gamma-1}(\omega_n^{-\gamma-1} r)^{-\gamma} e^{-\omega_n^{-\gamma-1} r/2} L_n^{-2\gamma-1}(\omega_n^{-\gamma-1} r) & , \quad \kappa < 0 \end{cases} \tag{3.10}$$

where $\omega_n^\gamma = 2\left|Z\varepsilon_n^\gamma\right|/(n+\gamma+1)$ and A_n^γ is the normalization constant $\sqrt{\frac{\omega_n^\gamma \Gamma(n+1)}{\Gamma(n+2\gamma+2)}}$. The lower spinor component is obtained by substituting this in Eq. (3.3) which, in this case, reads

$$\phi^-(r) = \frac{\lambda}{\varepsilon+\gamma/\kappa}\left(-\frac{Z}{\kappa}+\frac{\gamma}{r}+\frac{d}{dr}\right)\phi^+(r) \tag{3.11}$$

where $\varepsilon \neq -\gamma/\kappa$. Using the following recursion relations and differential formula satisfied by the Laguerre polynomials [15],

$$xL_n^\nu = (n+\nu)L_n^{\nu-1} - (n+1)L_{n+1}^{\nu-1} \tag{3.12a}$$

$$L_n^\nu = L_n^{\nu+1} - L_{n-1}^{\nu+1} \tag{3.12b}$$

$$-2\nu L_n^\nu + (n+\nu)L_n^{\nu-1} + (n+1)L_{n+1}^{\nu-1} = x\left(L_n^\nu - 2L_n^{\nu+1}\right) \tag{3.12c}$$

$$x\frac{d}{dx}L_n^\nu = nL_n^\nu - (n+\nu)L_{n-1}^\nu \tag{3.12d}$$

we obtain

$$\phi_n^-(r)=\frac{\lambda}{2}\begin{cases}\frac{\omega_n^\gamma-2Z/\kappa}{\varepsilon_n^\gamma+\gamma/\kappa}A_n^\gamma(\omega_n^\gamma r)^\gamma e^{-\omega_n^\gamma r/2}\left[(n+2\gamma+1)L_n^{2\gamma}(\omega_n^\gamma r)+\frac{\omega_n^\gamma+2Z/\kappa}{\omega_n^\gamma-2Z/\kappa}(n+1)L_{n+1}^{2\gamma}(\omega_n^\gamma r)\right] & ,\quad \kappa>0\\ -\frac{\omega_n^{-\gamma-1}-2Z/\kappa}{\varepsilon_n^{-\gamma-1}+\gamma/\kappa}A_n^{-\gamma-1}(\omega_n^{-\gamma-1}r)^{-\gamma}e^{-\omega_n^{-\gamma-1}r/2}\left[L_{n-1}^{-2\gamma}(\omega_n^{-\gamma-1}r)+\frac{\omega_n^{-\gamma-1}+2Z/\kappa}{\omega_n^{-\gamma-1}-2Z/\kappa}L_n^{-2\gamma}(\omega_n^{-\gamma-1}r)\right] & ,\quad \kappa<0\end{cases}$$

The spinor wavefunction associated with the lowest positive energy state is obtained from above, for $\kappa<0$, as

$$\psi_0(r)=\begin{pmatrix}\phi_0^+\\ \phi_0^-\end{pmatrix}=\sqrt{\frac{2Z/\kappa}{\Gamma(-2\gamma)}}(2Zr/\kappa)^{-\gamma}e^{-Zr/\kappa}\begin{pmatrix}1\\ -\lambda Z/\gamma\end{pmatrix} \tag{3.13}$$

where, for bound states, Z is negative. Obtaining the lower component of the spinor wavefunction associated with the highest negative energy state is more subtle. This is due to the fact that the "kinetic balance" relation (3.11) does not hold for this state (and only this state) since $\varepsilon=-\gamma/\kappa$. One has to redo the manipulations of Eq. (3.11) by considering carefully the limit $\varepsilon_0\to-\gamma/\kappa$ and for $n=0$.

B. Dirac superpotentials

In this case $V(r)=0$ and one does need to transform the radial Dirac equation (2.11) to obtain Schrödinger-like equations for the spinor components. In other words, the transformation parameter $\xi=0$ which corresponds to the identity transformation. The wave equations for the two radial spinor components obtained from (2.11) read as follows

$$\left[-\frac{d^2}{dr^2}+\frac{\kappa(\kappa\pm1)}{r^2}+2V_\kappa^\mp(r)-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi^\mp(r)=0 \tag{3.14}$$

where $V_\kappa^\pm=\frac{1}{2}\left(W^2\mp\frac{dW}{dr}\right)+\kappa\frac{W}{r}$. This equation could be written in a form which is more familiar in the language of supersymmetric quantum mechanics

$$\left[-\frac{d^2}{dr^2}+\left(U^2\pm\frac{dU}{dr}\right)-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi^\pm(r)=0 \tag{3.15}$$

where $U=W+\kappa/r$ and $U^2\pm U'$ are two superpartner potentials sharing the same energy spectrum (i.e., they are "isospectral") except for the lowest positive energy state and the highest negative energy state, where $\varepsilon=\pm1$, respectively. These two states belong only to U^2-U' [2]. Examples of problems that are associated with this case include the Dirac-Oscillator, where $W\sim r$. Dirac-Scarf, Dirac-Pöschl-Teller, and the Dirac-Rosen-Morse potentials. As an illustration, we consider the Dirac-Oscillator problem where $W(r)=\omega^2 r$, and ω is the oscillator frequency. This problem was formulated and solved by Moshinsky and Szczepaniak in their attempt to add to the Dirac equation a potential that is linear in the coordinate in an analogy to the kinetic energy which is linear in the momentum [7]. Eq. (3.14) with $W(r)=\omega^2 r$ gives

$$\left[-\frac{d^2}{dr^2}+\frac{\kappa(\kappa\pm1)}{r^2}+\omega^4r^2+\omega^2(2\kappa\mp1)-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi^\pm(r)=0 \tag{3.16}$$

We compare this equation with that for the nonrelativistic three dimensional isotropic oscillator:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+\omega^4r^2-2E\right]\Phi(r)=0 \tag{3.17}$$

The comparison gives the following two maps between the relativistic and nonrelativistic problems:

$$\text{for } \phi^+ : \ \omega \to \omega, E \to \frac{\varepsilon^2-1}{2\lambda^2} - \omega^2(\kappa - \tfrac{1}{2}), \ell \to \begin{cases} \kappa \\ -\kappa-1 \end{cases} \tag{3.18a}$$

$$\text{for } \phi^- : \ \omega \to \omega, E \to \frac{\varepsilon^2-1}{2\lambda^2} - \omega^2(\kappa + \tfrac{1}{2}), \ell \to \begin{cases} \kappa-1 \\ -\kappa \end{cases} \tag{3.18b}$$

Again, the top (bottom) choice of the ℓ map corresponds to positive (negative) values of κ; respectively. Using these parameter maps and the well-known nonrelativistic energy spectrum, $E_n = \omega^2(2n+\ell+3/2)$, we obtain the following relativistic spectrum

$$\varepsilon_n = \begin{cases} \pm\sqrt{1+4\lambda^2\omega^2(n+\kappa+\tfrac{1}{2})} & , \quad \kappa > 0 \\ \pm\sqrt{1+4\lambda^2\omega^2 n} & , \quad \kappa < 0 \end{cases}; \qquad n = 0,1,2,\ldots \tag{3.19}$$

The lowest/highest positive/negative energy state, where $\varepsilon = \pm 1$, respectively, occurs for $\kappa < 0$. It is associated with the spinor wavefunction $\psi = \begin{pmatrix} \phi^+ \\ 0 \end{pmatrix}_{\kappa<0}$. The radial components of the wavefnnction are obtained using the same parameter maps (3.18) and the non-relativistic wavefunction $\Phi_n(r) \sim (\omega r)^{\ell+1} e^{-\omega^2 r^2/2} L_n^{\ell+1/2}(\omega^2 r^2)$ giving for $n = 0,1,2,\ldots$

$$\psi_n(r) = (\omega r)^{\kappa} e^{-\omega^2 r^2/2} \begin{pmatrix} A_n^{\kappa}\, \omega r L_n^{\kappa+1/2}(\omega^2 r^2) \\ B_n^{\kappa} L_n^{\kappa-1/2}(\omega^2 r^2) \end{pmatrix}, \qquad \kappa > 0 \tag{3.20a}$$

$$\psi_n(r) = (\omega r)^{-\kappa} e^{-\omega^2 r^2/2} \begin{pmatrix} A_{n+1}^{-\kappa-1} L_{n+1}^{-\kappa-1/2}(\omega^2 r^2) \\ B_n^{-\kappa-1}\, \omega r L_n^{-\kappa+1/2}(\omega^2 r^2) \end{pmatrix}, \quad \kappa < 0 \tag{3.20b}$$

where A_n^{κ} and B_n^{κ} are normalization constant. They are related by the kinetic balance relation, $\phi^- = \frac{\lambda}{\varepsilon+1}\left(\omega^2 r + \frac{\kappa}{r} + \frac{d}{dr}\right)\phi^+$, which gives

$$B_n^{\kappa} = \frac{2\lambda\omega}{\varepsilon_n+1}(n+\kappa+\tfrac{1}{2})A_n^{\kappa} \tag{3.21}$$

The spinor wavefunction associated with lowest/highest positive/negative energy state, where $\varepsilon = \pm 1$ and $\kappa < 0$, is $\psi(r) = A_0^{-\kappa-1}(\omega r)^{-\kappa} e^{-\omega^2 r^2/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

C. The Dirac-Hulthén problem

The Hulthén potential is of considerable significance to various applications in many areas of physics. This includes applications in nuclear and particle physics, atomic physics, condensed matter, and chemical physics. The Dirac equation with this potential was solved for s-wave ($\ell = 0$) by Guo *et al.* [12], using the same approach presented above. In this case $V(r) = -A/(e^{\omega r}-1)$, where ω is the screening range of the potential. A and ω are real and positive. Equation (3.4) for the upper spinor components gives

$$\left[-\frac{d^2}{dr^2} + \frac{\rho(\rho+\omega)}{(e^{\omega r}-1)^2} + \frac{\rho\omega - 2\varepsilon A}{e^{\omega r}-1} - \frac{\varepsilon^2-1}{\lambda^2}\right]\phi^+(r) = 0 \tag{3.22}$$

where $\rho = \tau\sqrt{1-(\lambda A/\tau)^2}$ and $\tau = -A/\xi$. If we choose the transformation parameter ξ such that $\rho = -\omega$ then we obtain the solution found by Guo *et al.* in [12]. Nonetheless, we consider here the solution of an s-wave "generalized Dirac-Hulthén" problem where ρ is arbitrary and the potential consists of the sum of two terms: the Hulthén potential and its square. Taking $x = 1 - 2e^{-\omega r}$ maps real space into a bounded one. That is, $r \in [0, \infty] \to x \in [-1, +1]$. An L^2 function that is compatible with the domain of the wave operator and satisfies the boundary conditions in this configuration space could be written as

$$\phi_n^+(r) = a_n (1+x)^{\alpha} (1-x)^{\beta} P_n^{(\mu,\nu)}(x) \tag{3.23}$$

where the real parameters $\alpha, \beta > 0$ and $\mu, \nu > -1$. $P_n^{(\mu,\nu)}(x)$ is the Jacobi polynomial and a_n is the normalization constant $\sqrt{\frac{\omega(2n+\mu+\nu+1)}{2^{\mu+\nu+1}} \frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}}$. Using the differential equation, differential formulas of the Jacobi polynomial [15], and $\frac{d}{dr} = \omega(1-x)\frac{d}{dx}$ we can write

$$\begin{aligned} \frac{d^2\phi_n^+}{dr^2} = \omega^2 \frac{1-x}{1+x} \Big\{ \Big[& -n\left(x + \tfrac{\nu-\mu}{2n+\mu+\nu}\right)\left(\tfrac{\mu-2\beta}{1-x} + \tfrac{2\alpha-\nu-1}{1+x}\right) - n(n+\mu+\nu+1) - \alpha(2\beta+1) \\ & + \beta^2 \frac{1+x}{1-x} + \alpha(\alpha-1)\frac{1-x}{1+x} \Big] \phi_n^+ + 2\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu}\left(\frac{\mu-2\beta}{1-x} + \frac{2\alpha-\nu-1}{1+x}\right)\frac{a_n}{a_{n-1}}\phi_{n-1}^+ \Big\} \end{aligned} \tag{3.24}$$

To eliminate the off-diagonal representation (the ϕ_{n-1}^+ term) we should take $\mu = 2\beta$ and $\nu = 2\alpha - 1$. Consequently, we obtain

$$\left\{ -\frac{d^2}{dr^2} + \omega^2\alpha(\alpha-1)\left(\tfrac{1-x}{1+x}\right)^2 - \omega^2\left[n(n+2\alpha+\mu) + \alpha(\mu+1)\right]\tfrac{1-x}{1+x} + \frac{\mu^2\omega^2}{4} \right\} \phi_n^+ = 0 \tag{3.25}$$

Comparing this with Eq. (3.22) gives the following parameter assignment

$$\alpha = \begin{cases} 1+\zeta & ,\tau \ge 0 \\ -\zeta & ,\tau < 0 \end{cases} \quad , \quad \mu_n = \begin{cases} \lambda_{n+1}^{\zeta} & ,\tau \ge 0 \\ \lambda_n^{-\zeta} & ,\tau < 0 \end{cases} \tag{3.26}$$

where $\zeta \equiv \rho/\omega$ and $\lambda_n^{\zeta} = -(n+\zeta) + (n+\zeta)^{-1}\left(\zeta^2 + 2\varepsilon A/\omega^2\right)$. The energy spectrum is obtained as the two real solutions, $\varepsilon_n^{\pm}$, of the following quadratic (in ε) parameter relation which also results form the comparison of the two equations

$$\varepsilon^2 = 1 - (\lambda\omega/2)^2 \mu_n^2 \quad , \ n = 0, 1, 2, \ldots, n_{max} \tag{3.27}$$

$\pm\varepsilon_n^{\pm} > 0$ and n_{max} is the maximum integer n that yields real solutions for Eq. (3.27). The lowest positive energy eigenvalue is $\varepsilon_{n_{max}}^+$, while the highest negative energy in the spectrum is $\varepsilon_{n_{max}}^-$. Furthermore, it is easy to verify that for n = 0 and when $\tau < 0$ Eq. (3.27) gives $\varepsilon_0^{\pm} = \pm\rho/\tau$. The four energy bounds $\left\{\varepsilon_0^{\pm}\big|_{\tau<0}, \varepsilon_{n_{max}}^{\pm}\big|_{\tau>0}\right\}$ correspond to non-degenerate states, while all others do not. This is because $\varepsilon_n^{\pm}\big|_{\tau>0} = \varepsilon_{n+1}^{\pm}\big|_{\tau<0}$ for $n = 0, 1, 2, \ldots, n_{max} - 1$ and for all τ. Figure 1 shows the relativistic energy spectrum for a given set of physical parameters and for several values of the dimensionless parameter ζ.

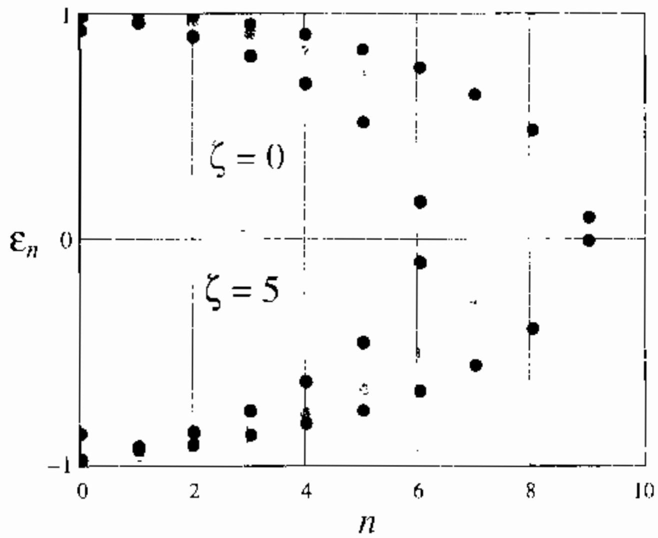

Fig. 1: The relativistic energy spectrum of the s-wave "generalized Dirac-Hulthén" problem for several values of the dimensionless parameter ζ. The other physical parameters are taken (in arbitrary units) as $A = 2.5, \omega = 1.0, \lambda = 0.2$. The graph also shows that in this example $n_{max} = 6, 7, 9$ for $\zeta = 5, -4, 0$, respectively.

Now, the upper radial spinor component is obtained by substituting the parameters of (3.26) and (3.27) into the wavefunction (3.23) giving

$$\phi_n^+(r) = \begin{cases} a_n^{\zeta+1}\sqrt{2^{\lambda_{n-1}^{\zeta}}}\, e^{-\omega\lambda_{n-1}^{\zeta} r/2}(1-e^{-\omega r})^{\zeta+1} P_n^{(\lambda_{n-1}^{\zeta},\, 2\zeta+1)}(1-2e^{-\omega r}) & , \quad \tau \geq 0 \\ a_n^{-\zeta}\sqrt{2^{\lambda_n^{-\zeta}}}\, e^{-\omega\lambda_n^{-\zeta} r/2}(1-e^{-\omega r})^{-\zeta} P_n^{(\lambda_n^{-\zeta},\, -2\zeta-1)}(1-2e^{-\omega r}) & , \quad \tau < 0 \end{cases} \tag{3.28}$$

The lower component of the spinor wavefunction is obtained by substituting this in Eq. (3.3) which, in the x-coordinate, reads as follows

$$\phi^-(r) = \frac{\lambda\omega}{\varepsilon + \rho/\tau}\left[\frac{A}{\tau\omega} + \zeta\left(\frac{1-x}{1+x}\right) + (1-x)\frac{d}{dx}\right]\phi^+(r) \tag{3.29}$$

where $\varepsilon \neq -\rho/\tau$. Using the following recursion relations and differential formula satisfied by the Jacobi polynomials [15],

$$\left(\frac{1+x}{2}\right) P_n^{(\mu,\nu)} = \frac{n+\nu}{2n+\mu+\nu+1} P_n^{(\mu,\nu-1)} + \frac{n+1}{2n+\mu+\nu+1} P_{n+1}^{(\mu,\nu-1)} \tag{3.30a}$$

$$P_n^{(\mu,\nu)} = \frac{n+\mu+\nu+1}{2n+\mu+\nu+1} P_n^{(\mu,\nu+1)} + \frac{n+\mu}{2n+\mu+\nu+1} P_{n-1}^{(\mu,\nu+1)} \tag{3.30b}$$

$$\left(1-x^2\right)\frac{d}{dx} P_n^{(\mu,\nu)} = -n\left(x + \frac{\nu-\mu}{2n+\mu+\nu}\right) P_n^{(\mu,\nu)} + 2\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu} P_{n-1}^{(\mu,\nu)} \tag{3.30c}$$

we obtain the following expressions for the lower spinor component when $\tau \geq 0$

$$\begin{aligned} \phi_n^-(r) = \frac{\lambda\omega a_n^{\zeta+1}}{\varepsilon_{n-1}^{\zeta}+\rho/\tau}\sqrt{2^{\lambda_{n-1}^{\zeta}}}\, e^{-\omega\lambda_{n-1}^{\zeta} r/2}(1-e^{-\omega r})^{\zeta}\Bigg[\left(\frac{-\zeta+A/\tau\omega}{n+1+\zeta+\lambda_{n-1}^{\zeta}/2}+1\right)(n+2\zeta+1)\times \\ P_n^{(\lambda_{n-1}^{\zeta},\, 2\zeta)}(1-2e^{-\omega r}) + \left(\frac{-\zeta+A/\tau\omega}{n+1+\zeta+\lambda_{n-1}^{\zeta}/2}-1\right)(n+1) P_{n+1}^{(\lambda_{n-1}^{\zeta},\, 2\zeta)}(1-2e^{-\omega r})\Bigg] \end{aligned} \tag{3.31a}$$

while for $\tau < 0$, the result is

$$\phi_n^-(r) = -\frac{2\lambda\omega a_n^{-\zeta}}{\varepsilon_n^{\zeta}+\rho/\tau}\sqrt{2^{\lambda_n^{-\zeta}}}\,e^{-\omega\lambda_n^{-\zeta}r/2}(1-e^{-\omega r})^{-\zeta}\Big\{\big(n+\lambda_n^{-\zeta}/2-A/\tau\omega\big)P_n^{(\lambda_n^{-\zeta},\,-2\zeta-1)}(1-2e^{-\omega r})+ \\ -(1-e^{-\omega r})^{-1}\Big[(2\zeta+1)P_n^{(\lambda_n^{-\zeta},\,-2\zeta-1)}(1-2e^{-\omega r})+(n-2\zeta-1)P_{n+1}^{(\lambda_n^{-\zeta},\,-2\zeta-2)}(1-2e^{-\omega r})\Big]\Big\} \quad (3.31b)$$

Using the Jacobi polynomial identity $P_n^{(\mu,\nu)}(-1)=(-)^n\frac{\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(\nu+1)}$ one can verify that the term inside square hrackets in (3.31h) vanishes at $r=0$. Moreover, one can also show that

$$\lim_{\delta\to 0}\Big[\nu P_n^{(\mu,\nu)}(-1+\delta)-(n+\nu)P_n^{(\mu,\nu-1)}(-1+\delta)\Big]=\frac{(-)^n\Gamma(n+\nu+1)}{2\Gamma(n)\Gamma(\nu+2)}(n+\mu)\;\delta+O(\delta^2) \quad (3.32)$$

Therefore, the term with the $(1-e^{-\omega r})^{-1}$ factor in (3.31b) is finite at r = 0 maintaining square integrability of $\phi_n^-(r)$.

Table I: A list of all relativistic problems that have been solved up to date by the approach presented in this paper. For each problem the Table lists the even and odd potential components, the transformation parameter in terms of the potential parameters, and any relation among the parameters.

	$V(r)$	$W(r)$	ξ	Parameters Relation
Dirac-Coulomb	Z/r	0	Z/κ	$\kappa^2=\gamma^2+\lambda^2Z^2$
Dirac-Oscillator	0	$\omega^2 r$	0	———
Dirac-Morse	$-Ae^{-\omega r}$	$-\tau e^{-\omega r}-\kappa/r$	A/τ	$\tau^2=\omega^2+\lambda^2A^2$
Dirac-Rosen-Morse I	$A\tanh(\omega r)$	$\tau\tanh(\omega r)-\kappa/r$	A/τ	$\tau^2=\omega^2+\lambda^2A^2$
Dirac-Eckart	$A\coth(\omega r)$	$\tau\coth(\omega r)-\kappa/r$	A/τ	$\tau^2=\omega^2+\lambda^2A^2$
Dirac-Rosen-Morse II	0	$A\coth(\omega r)-B\operatorname{csch}(\omega r)-\kappa/$	0	———
Dirac-Scarf	0	$A\tanh(\omega r)+B\operatorname{sech}(\omega r)-\kappa/$	0	———
Dirac-Pöschl-Teller	0	$A\tanh(\omega r)+B\coth(\omega r)-\kappa/r$	0	———
Dirac-Woods-Saxon	$\frac{-A}{1+e^{\omega(r-R)}}$	$\frac{\tau}{1+e^{\omega(r-R)}}-\kappa/r$	$-A/\tau$	$\tau^2=\omega^2+\lambda^2A^2$
Dirac-Hulthén	$\frac{-A}{e^{\omega r}-1}$	$\frac{\tau}{e^{\omega r}-1}-\kappa/r$	$-A/\tau$	$\tau^2=\omega^2+\lambda^2A^2$

Table I lists all relativistic problems that have heen solved by this approach up to date. For each problem the table lists the even and odd potential components, the transformation parameter in terms of the potential parameters, and any relation among the parameters. A comprehensive list of all potential classes and the "extended point canonical transformations" that map all potentials in a given class into one another are given in Ref. [16].

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حلول جديدة لمعادلة "ديراك"

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المستخلص. نعرض في هذه الورقة طريقة طوّرناها حديثاً لحل معادلة "ديراك" التي تصف ديناميكية الأجسام ذات الأبعاد الذرية والسرعات العالية المقاربة لسرعة الضوء والتي يكون دورانها الذاتي نصف عددي . لقد تمكنّا باستعمال هذه الطريقة من الحصول على حلول جديدة لمعادلة "ديراك" لعدد من التفاعلات في حيّز الأبعاد الثلاثية المعتادة شريطة أن يكون التفاعل ذا تناسق مركزي كروي ، كما أننا قد أثبتنا أنه عند انخفاض السرعات عن مستوياتها في النظرية النسبية فإننا نحصل عندها على الحلول التقليدية المعهودة في نظرية الكم التي تصفها معادلات مثل معادلة "شرودينقر" .

إن الفكرة الأساسية في هذه الطريقة المُستحدثة هي تحويل معادلة "ديراك" بحيث يمكن فكّ الترابط بين المعادلتين لجزئي الجسم الذري التي نحصل عليها عادة من معادلة "ديراك" إلى معادلتين منفصلتين تتشابه بدرجة كبيرة مع معادلات نعرف حلولها مُسبقاً ، ثم نستخدم هذا التشابه في عمل خارطة أو جدول يساعدنا على تحويل الحلول للمعادلات المعروفة إلى حلول لمعادلة "ديراك" ، وبذا فإنه يمكننا الحصول على حل لمعادلة "ديراك" بسهولة ويسر بمجرّد استخدام هذا الجدول أو الخارطة ، ولقد استعملت هذه الطريقة من قبل المؤلف وغيره ممن استحسنوها للحصول على العديد من الحلول الجديدة لمعادلة "ديراك" .